Unification of terrestrial and hydrographic refractions: new model in the case of a vertically stratified velocity field on a locally spherical Earth

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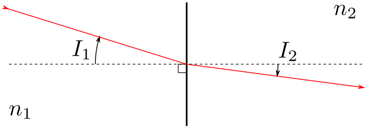
Refraction appears when a wave, that is propagating through a media, is slowed by the environmental changes. It affects electromagnetic waves in the air, in land surveying, as well as acoustic beams in hydrography.
Both of these communities have independently developed methods to take into account refraction in the observations processing.
In this presentation, we will show that refraction in both cases of land surveying and hydrography can be unified in the same model that takes benefit to both of their usual methods. After the introduction of the new model, we will present results from hydrographic surveys simulations.

A huge amount of mathematics is behind this presentation. Please try not to focus on formulae but rather on the phenomena we try to model.

Discrete case

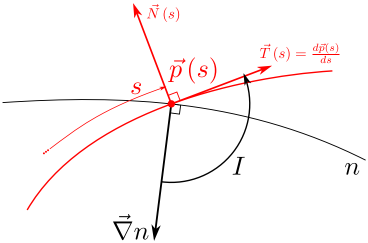
A diopter splits space into two different refractive media.



$$n \sin I = \text{constant}$$

$$\rightarrow \frac{\partial}{\partial s} \rightarrow$$

Continuous case



$$\frac{dI}{ds} = -\frac{\|\vec{\nabla} n\|}{n} \sin I$$

See [Torge, 2001] or [Touzé, 2022a]. For hydrographic applications, one can use the velocity v instead of the refractive index $n = \frac{c}{v}$.

Model deduced from Fermat principle or from the propagation of an electro-magnetic wave in a refractive media. **One can show that Snell-Descartes implies the eikonal equation** [Touzé, 2022a].

$$\frac{d}{ds}\left[n\,\frac{d\vec{p}}{ds}\right] = \vec{\nabla} n$$

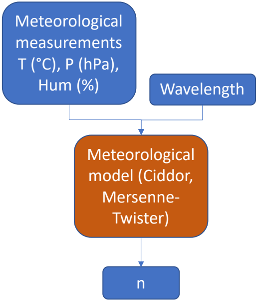
It can be rewritten in a useful way for ray-tracing [Touzé, 2022a] :

$$\frac{d^2\vec{p}}{ds^2} = \frac{\vec{\nabla} n}{n} - \left(\frac{\vec{\nabla} n}{n} \cdot \frac{d\vec{p}}{ds}\right)\frac{d\vec{p}}{ds}$$

And with velocity v instead of n :

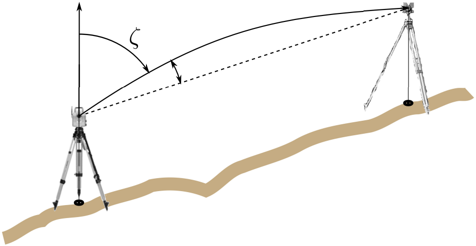
$$\frac{d^2\vec{p}}{ds^2} = \left(\frac{\vec{\nabla} v}{v} \cdot \frac{d\vec{p}}{ds}\right)\frac{d\vec{p}}{ds} - \frac{\vec{\nabla} v}{v}$$

Distances calibrations



Knowing n, the distance can be computed by the device from time delays or phases measurements

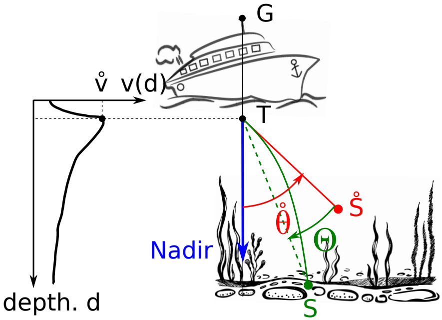
Angular deflection



Difference of altitude ΔH takes into account the angular deflection by introducing the refraction coefficient k such as :

$$\Delta H = D \cos \zeta + (1 - k) \frac{D^2 \sin^2 \zeta}{2R}$$

Ray-tracing from Snell-Descartes through a velocity field



Initial conditions :

- Position
- Tangential direction at the transductor

Knowing the time delay and the velocity profile, by applying snell-Descartes law at an infinitesimal step, one can deduce the position of the sounding

State of the art in refraction

A bridge has to be built

Strength of terrestrial refraction

- Refraction modeled by a scale factor and an angular deflection

Strength in hydrography

- Velocity profiles used in ray-tracing
- accurate if enough SVP

Issues on terrestrial refraction

- Standard model of $k \approx 0.13$ is seldom accurate
- Relationship between k refraction coefficient and meteorological data is not clearly defined

Issues in hydrography

- Ray-tracing makes difficult the quality checks, in overlapping area, of velocity profiles
- *As far as I know*, the effect of the velocity gradient is not taken into account

Both terrestrial and hydrographic approaches can improve each other !

Hypothesis

- $\mathcal{H}_1$: v velocity field of class $\mathcal{C}_1$ (continuous & differentiable)
- $\mathcal{H}_2$: Equipotential surfaces of gravity are locally spherical & concentric (Earth locally spherical)
- $\mathcal{H}_3$: v velocity field is only varying with altitude H

Coordinates problem

- We want to integrate eikonal equation (quite easy in an Euclidean space)
- But it depends on the gradient of v that is curved like the Earth

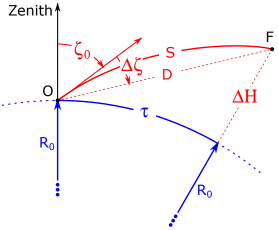
$\Longrightarrow$ We will use tensors from absolute differential geometry to rewrite the eikonal equation and solve it in curved space (See Eigenchris' *Youtube* channel with "Tensors for beginner" and "Tensors calculus" series)

Using Einstein's summation convention and $\mathcal{H}_1$: [Touzé, 2022a]

$$\frac{d^2x^k}{ds^2} = \frac{1}{v}\frac{\partial v}{\partial x^i}\left(\frac{dx^i}{ds}\frac{dx^k}{ds} - g^{ik}\right) - \Gamma^k_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}$$

With g^{ij} : contravariant metric tensor and Γ^k_{ij} : Christoffel symbols

Adding $\mathcal{H}_2$ & $\mathcal{H}_3$ + (quite a lot of) calculus : [Touzé, 2022a]



$$\begin{cases} \dfrac{d^2H}{ds^2} \approx \dfrac{\sin^2\zeta_0}{v_0^2}\left(\dfrac{v^2}{R_0} - v\,v'\right) \\ \dfrac{d^2\tau}{ds^2} \approx \dfrac{\sin\zeta_0}{v_0}\left(v' - \dfrac{2v}{R_0}\right)(\cos\zeta_0 - \text{correction}) \end{cases}$$

with $v' = \frac{dv}{dH}$ & v_0 : velocity at emission (SVS)

Velocity field

Along the path, we known p discrete values v_i of the velocity at constant length interval. One can deduce the p values v_i' of the gradient.

Length of the path S

With round trip duration Δt, we look for the true mean velocity $\hat{v}$ such as

$S = \hat{v}\,\frac{\Delta t}{2}$ We have :

$$\frac{1}{\hat{v}} = \frac{1}{S}\int_0^S \frac{1}{v}\,ds \approx \frac{1}{p}\sum_{i=1}^{p}\frac{1}{v_i}$$

Cauchy formula for repeated integration + initial conditions

$$\Delta H = S\cos\zeta_0 + \int_0^S (S-s)\,\frac{d^2H}{ds^2}\,ds$$

And a similar expression for $\tau(S)$

Let's introduce all these parameters [Touzé, 2022b]

$$\hat{w} = \frac{1}{S}\int_0^S v\,ds \approx \frac{1}{p}\sum_{i=1}^{p} v_i \qquad \rho = \frac{\displaystyle\int_0^S s\,v\,ds}{S\displaystyle\int_0^S v\,ds} \approx \frac{\sum s_i\,v_i}{S\sum v_i}$$

$$\hat{w}' = \frac{1}{S}\int_0^S v'\,ds \approx \frac{1}{p}\sum_{i=1}^{p} v'_i \qquad \rho' = \frac{\displaystyle\int_0^S s\,v'\,ds}{S\displaystyle\int_0^S v'\,ds} \approx \frac{\sum s_i\,v'_i}{S\sum v'_i}$$

$$\Delta v = v - \hat{w} \qquad \Delta v' = v' - \hat{w}'$$

$$\sigma_w^2 = \frac{1}{S}\int_0^S \Delta v^2\,ds \approx \frac{1}{p}\sum_{i=1}^{p} \Delta {v_i}^2 \qquad \rho_{ww} = \frac{\displaystyle\int_0^S s\,\Delta v^2\,ds}{S\displaystyle\int_0^S \Delta v^2\,ds} \approx \frac{\sum s_i\,\Delta {v_i}^2}{S\sum \Delta {v_i}^2}$$

$$\sigma_{ww'} = \frac{1}{S}\int_0^S \Delta v \Delta v'\,ds \approx \frac{1}{p}\sum_{i=1}^{p} \Delta v_i \Delta v'_i \qquad \rho_{ww'} = \frac{\displaystyle\int_0^S s\,\Delta v \Delta v'\,ds}{S\displaystyle\int_0^S \Delta v \Delta v'\,ds} \approx \frac{\sum s_i\,\Delta v_i \Delta v'_i}{S\sum \Delta v_i \Delta v'_i}$$

$$\mu = (1-\rho_{ww})\,\frac{\sigma_w^2}{\hat{w}^2} \qquad \mu' = (1-\rho_{ww'})\,\frac{\sigma_{ww'}}{\hat{w}\hat{w}'}$$

Believe me or not, but these horrible formulas are very easy to deduce from velocity data (means, weighted means, covariance and standard deviation) !

We can also introduce the refraction coefficient k :

$$k = \frac{R_0\,\hat{w}'}{\hat{w}}$$

1. From velocity profile, compute $\hat{v}$, $\hat{w}$, $\hat{w}'$, ρ, ρ', μ, μ' and k
2. Length of the path $S = \hat{v}\,\frac{\Delta t}{2}$
3. Difference of altitude $\Delta H = S\cos\zeta_0 + \frac{S^2\sin^2\zeta_0}{2R_0}\,\frac{\hat{w}^2}{v_0^2}\,(A - k\,B)$ with $A = 3 - 4\rho + 2\mu$ and $B = 3 - 2\rho - 2\rho' + 2\mu'$
4. Arch length $\tau = S\sin\zeta_0 - \frac{S^2\sin\zeta_0\cos\zeta_0}{R_0}\,\frac{\hat{w}}{v_0}\,(2\,(1-\rho) - k\,(1-\rho')) - (\text{correction})$
5. Euclidean distance $D = \sqrt{\Delta H^2 + \left(1 + \frac{\Delta H}{R_0}\right)\tau^2}$
6. Angular correction $\Delta\zeta = \operatorname{acos}\left(\frac{\Delta H}{D}\left(1 + \frac{\Delta H}{2R_0}\right) - \frac{D}{2R_0}\right) - \zeta_0$

Now, refraction can be considered as a rotation of the initial tangent $\vec{T}_0$, by angle $\Delta\zeta$ around the vector $\overrightarrow{\text{zenith}} \times \vec{T}_0$, and scaled by distance D.

Simulations from true velocity profiles

Method of comparison

Method

- 2 measured SVP (60 m deep fresh water & 2'300 m deep sea water)
- 400 simulated beams (nadir angle up to 88°) with eikonal ray-tracing
- From emitted angles and round trip durations, new method applied
- Formulae modifications required for vertical sound velocity profiles

For vertical velocity profiles

- p velocity v_i and gradient v_i' are known at constant depth step
- $\cos\zeta_i = \pm\sqrt{1 - \frac{v_i^2}{v_0^2}\sin^2\zeta_0} \approx \cos\zeta_0 - \frac{\sin^2\zeta_0}{v_0^2}\int_0^s v\,v'\,ds$
- Formulae for $\hat{v}$, $\hat{w}$, $\hat{w}'$, ρ and ρ' must be modified, for instance :

$$\hat{w} \approx \frac{\sum \frac{v_i}{\cos\zeta_i}}{\sum \frac{1}{\cos\zeta_i}}$$

Process

1. Initialization with $S = v_0 \frac{\Delta t}{2}$ & $\Delta\zeta = 0 \rightarrow$ approx sounding $\mathring{F}$
2. Extraction, from the SVP, of the velocity and gradient values v_i & v_i'
3. Computing $\hat{v}$, $\hat{w}$, $\hat{w}'$, ρ, ρ', μ, μ' and k
4. Then ΔH, τ, D and $\Delta\zeta \rightarrow$ New sounding F
5. While $\|\mathring{F}F\| \geq \varepsilon$ go to step 2 with $\mathring{F} := F$
6. End $\rightarrow$ Final sounding F

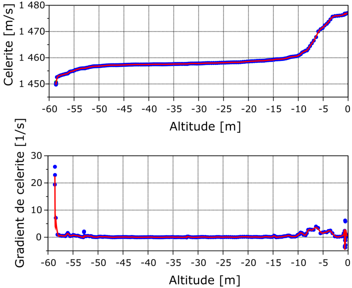
Comments on the process

- Most of the time, 5 iterations were required for 10^{-6} convergence
- Let's compare the 3D distances between our final soundings and ray-tracing !

Simulations from true velocity profiles

Results in 60 m deep fresh water

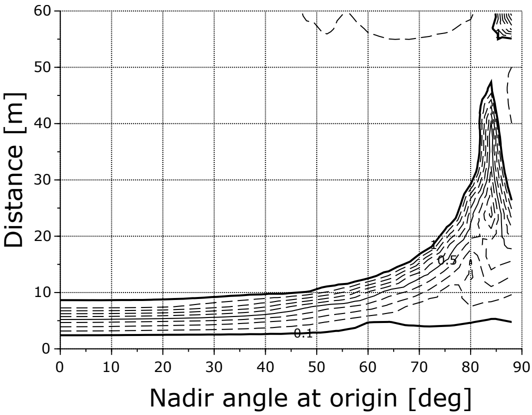
Velocity & gradient



Measured by GEOxyz in Monteynard lake (FR) (thanks !)

Comparisons with 20 cm step ray-tracing

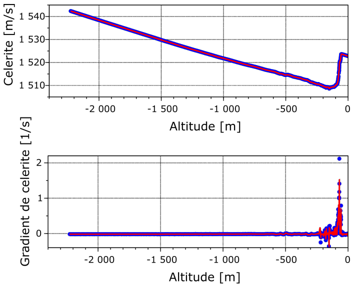
3D differences in mm :



Simulations from true velocity profiles

Results in 2'300 m deep sea water

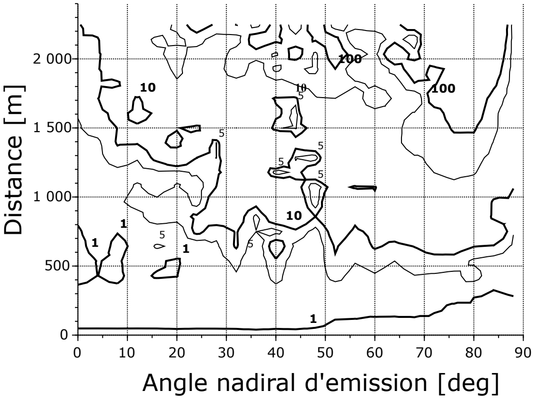
Velocity & gradient



Measured by iXBlue in Mediterranean sea (thanks !)

Comparisons with 50 cm step ray-tracing

3D differences in mm :



Summary of the results

The 3D distances between ray-tracing and our model are smaller than :

- 2 mm in 60 m deep fresh water (20 cm step for ray tracing)
- 5 cm in 2'300 m deep sea water (50 cm step) for distances smaller than 1'500 m or angle smaller than 65°

Comments

- Promising results, but it requires further investigations with true observations
- In land surveying, relative variations of velocity are 1'000 times smaller. Hence we expect 1'000 times better results !

Conclusion

- New method takes benefit from both ways, in land surveying and hydrography, to deal with refraction in a unified model
- It seems able to solve all the weak points of slide 8
- This model, in hydrography, allows much easier velocity errors analysis
- Further validations are required but the first results are promising
- Refraction can now be written as a rotation : it can also be use in photogrammetry
- I'm currently writing an english article describing entirely this work, it should be publish in 2023

References

TORGE, W. (2001).
Geodesy, third completely revised and extended edition.
Walter de Gruyter.

TOUZÉ, T. (2022a).
Généralisation des réfractions terrestres et bathymétriques : nouveau modèle dans le cas d'une stratification verticale de la célérité en terre localement sphérique.
XYZ, (170).

TOUZÉ, T. (2022b).
Généralisation des réfractions terrestres et bathymétriques : utilisation concrète et simulations de levés bathymétriques au sonar multifaisceaux.
XYZ, (172).